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Non-local ansätze for nonlinear heat and wave equations

Ivan Tsyfra

Institute of Mathematics of the National Ukrainian Academy of Sciences, Tereshchenkivska Street 3, 252004 Kyiv, Ukraina

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Abstract. New ansätze reducing nonlinear heat and wave equations to a system of ordinary differential equations are proposed. These ansätze can be constructed with the help of symmetry operators of the system corresponding to the equation under study. Exact solutions of the considered equations are obtained. The link between the Bäcklund transformations for a given equation and the conditional symmetry of the corresponding system is discussed.

The papers [1, 2] are devoted to construction of exact solutions of nonlinear differential equations by using the ansatz

$$u = F(x, \varphi(\omega)) \qquad \omega = \omega(x) \tag{1}$$

which reduces partial differential equations to equations with a smaller number of independent variables (for more details of the reduction of partial differential equations see also [3, 6, 8–15]. By means of (1), exact solutions of many linear and nonlinear mathematical physics equations have been found (see [3] and references therein). It is well known that the classical Lie method of infinitesimal transformations, the non-classical method [8], the method of conditional symmetries [3, 15], and the direct method [11] allow us to construct the ansatz of type (1) for a dependent variable u.

The technique for constructing the ansatz for the derivatives u_{x_i}

$$\frac{\partial u}{\partial x_i} = R_i(x, u, \varphi_1(\omega), \ldots, \varphi_n(\omega)) \qquad \omega = \omega(x, u) \qquad i = 1, 2, \ldots, n \tag{2}$$

is discussed which reduces partial differential equations to the system of equations for $\varphi_1, \ldots, \varphi_n$ with a smaller number of independent variables. We consider (2) as a non-local ansatz for u because determination of an explicit solution of considered equation requires integration of a system of differential equations (2). It is obvious that the compatibility condition

$$\frac{\partial R_k}{\partial x_l} = \frac{\partial R_l}{\partial x_k}$$

has to be satisfied in this case. To find R_0 and R_k we use the idea proposed in [4]. The essence of this idea is to replace the second-order partial differential equation for a scalar function by a system of two first-order equations and then study the symmetry of this system. Indeed, in this way a new invariance algebra of the system, which is equivalent to the Klein–Gordon–Fock equation has been constructed in [4]. An analogous idea was applied by Bluman and Kumei to investigation of the symmetry of the wave equation [9, 10].

It should be noted that in the general case the symmetry group of the corresponding system contains the symmetry group of the initial equation as a subgroup. With the help of the extended operators of point symmetry admitted by the considered equation we can construct ansätze of type (2), but they lead to the solutions of the given equation which can be obtained as invariant solutions of its point symmetry. Thus, to obtain new solutions it is necessary to use the operators of the classical point symmetry of the corresponding system which are not the extended operators of point symmetries admitted by the original equation as well as the operators of the conditional symmetry of the system.

One can consider this approach as a group method for construction of differential constraints [13, 16, 17], which are necessarily compatible, provided that the non-trivial solutions of the reduced system exist.

In [18] Galaktionov proposed the method of nonlinear separation to find the solutions to nonlinear diffusion equations. We show that the Galaktionov's ansatz can sometimes be obtained within the framework of this approach.

Another important application of this technique is its use in the construction of the Bäcklund transformations of partial differential equations. We briefly discuss the relationship between the conditional symmetry of the corresponding system and the Bäcklund transformations of the original equation.

As to the main ideas of the suggested approach, the present paper is close to [7].

(1) Let us consider the nonlinear equation

$$u_t = F(u_{xx}) \tag{3}$$

where $F(u_{xx})$ is a smooth function of its argument. This equation and the nonlinear heat equation

$$w_t - (c(w)w_x)_x = 0 \tag{4}$$

where $c(w) = \mathrm{d}F(w)/\mathrm{d}w$ are linked by the transformation $w = u_{xx}$. According to [4, 5], the following system

$$v^1_2 + v^1_3 v^2 = v^2_1 + v^2_3 v^1 \tag{5}$$

$$v^2_2 + v^2_3 v^2 = \Phi(v^1) \tag{6}$$

corresponds to equation (3), where $t \equiv x_1$, $x \equiv x_2$, $u \equiv x_3$, $\partial u/\partial t \equiv v^1$, $\partial u/\partial x \equiv v^2$, $v^i_k \equiv \partial v^i/\partial x_k$, and $\Phi(v^1) = F^{-1}(v^1)$.

In the general case, v^1 and v^2 are functions of variables x_1, x_2, x_3 and system (5) and (6) is not equivalent to equation (3). Nevertheless, it is invariant with respect to the group admitted by (3). Let us consider the operator

$$X = \xi^1\partial_{x_1} + \xi^2\partial_{x_2} + \xi^3\partial_{x_3} + \eta^1\partial_{v^1} + \eta^2\partial_{v^2} \tag{7}$$

where ξ^i and η^k, $i = 1, 2, 3$, $k = 1, 2$, are functions of x_1, x_2, x_3, v^1, v^2. It turns out that the invariance condition for equation (5)

$$\underset{1}{X}(v^1_2 + v^1_3 v^2 = v^2_1 + v^2_3 v^1)|_{v^1_2+v^1_3v^2=v^2_1+v^2_3v^1} \equiv 0 \tag{8}$$

where $\underset{1}{X}$ denotes the first extension of the infinitesimal operator X, is equivalent to the requirement of the invariance of the contact condition under group transformations and leads to the representations for η^1 and η^2 which are identical to the classic formulae for the extended infinitesimals η^{u_t} and η^{u_x} arising in the theory of prolongation [3, 10, 12, 14]. Requiring equation (6) to admit the operator X, and taking into account the fact that $x_1 \equiv t$, $x_2 \equiv x$, $x_3 \equiv u$, $v^1 \equiv \partial u/\partial t$, $v^2 \equiv \partial u/\partial x$ we obtain the Lie algebra which is the same as the algebra of infinitesimal operators of the first extended symmetry group of the initial equation (3).

If we study the symmetry properties of system (5) and (6) then (8) is changed and takes the form

$$\underset{1}{X}(v_2^1 + v_3^1 v^2 = v_1^2 + v_3^2 v^1)|_{\substack{v_2^1+v_3^1v^2=v_1^2+v_3^2v^1 \\ v_2^2+v_3^2v^2=\Phi(v^1)}} \equiv 0. \tag{9}$$

This means that condition (8) has to be satisfied not in all $(x_1, x_2, x_3, v^k, v_m^l)$-space but only on the manifold given by (6). It is obvious that condition (9) is weaker than condition (8). Hence there is the possibility of expansion of the class of symmetry operators.

Suppose system (5) and (6) admits a one-parameter group of transformations with infinitesimal generator of the form (7). Let $\omega_i(x_1, x_2, x_3, v^1, v^2)$, where $i = 1, 2, 3, 4$, be functionally independent invariants of (7) satisfying

$$X\omega_i = 0$$

and $\mathrm{rank}[\partial\omega_l/\partial v^k] = 2$, where $k = 1, 2$, $l = 3, 4$. Then we can construct the ansatz

$$\omega_3 = \varphi^1(\omega_1, \omega_2) \qquad \omega_4 = \varphi^2(\omega_1, \omega_2) \tag{10}$$

reducing system (5) and (6) to the system of two partial differential equations for φ^1 and φ^2 with two independent variables ω_1 and ω_2,

$$\begin{aligned} &L_1(\omega_1, \omega_2, \varphi^1, \varphi^1, \varphi^1_{\omega_1}, \varphi^1_{\omega_2}, \varphi^2_{\omega_1}, \varphi^2_{\omega_2}) = 0 \\ &L_2(\omega_1, \omega_2, \varphi^1, \varphi^1, \varphi^1_{\omega_1}, \varphi^1_{\omega_2}, \varphi^2_{\omega_1}, \varphi^2_{\omega_2}) = 0 \end{aligned} \tag{11}$$

where L_1 and L_2 are some functions of their arguments. In general, this system is equivalent to the initial equation (3). One can construct the solution $u(t, x)$ of the initial equation from the solution $\varphi^1(\omega_1, \omega_2)$ and $\varphi^2(\omega_1, \omega_2)$ of (11) by integrating the following system:

$$\begin{aligned} &\omega_3(t, x, u, u_t, u_x) = \varphi_1(\omega_1(t, x, u, u_t, u_x), \omega_2(t, x, u, u_t, u_x)) \\ &\omega_4(t, x, u, u_t, u_x) = \varphi_2(\omega_1(t, x, u, u_t, u_x), \omega_3(t, x, u, u_t, u_x)). \end{aligned} \tag{12}$$

Studying the symmetry of the reduced system (11) we can obtain the additional symmetry operators. These operators are used to reduce (11) to a system of ordinary differential equations. In a similar way we find the solutions of the original equation from the solutions $\varphi^1(\omega_1)$ and $\varphi^2(\omega_1)$ of the ordinary differential equations (see above (12)).

It should be emphasized that the extended operators of conditional symmetry admitted by (3) generate the ansätze which do not reduce this equation. Therefore, we use the operators of conditional symmetry admitted by system (5) and (6) (see below (2), (3) and (6)).

Note that using one symmetry operator we construct the ansatz which reduces (5) and (6) to the system of partial differential equations with two independent variables. In particular, when $\xi^1 = \xi^2 = 0$ in (7) we obtain the reduced system for functions $\varphi^1(t, x)$ and $\varphi^2(t, x)$ depending on two variables t and x. In this case the ansatz can be used for constructing a Bäcklund transformation for the original equation (see below (6)). A two-dimensional algebra $\{Q_1, Q_2\}$ which corresponds to the two-parameter group having three independent invariants leads to the system of ordinary differential equations. Thus the number of independent variables of the reduced system obtained from (5) and (6) equals $k - 2$, where k is the number of functionally independent invariants of the Lie group being used for construction of the ansätze.

Taking into account the invariance of equation (3) with respect to the translation group $u' = u + a$, where a is a parameter, we can reduce (5) and (6) to the system

$$v_2^1 = v_1^2 \qquad v_2^2 = \Phi(v^1). \tag{13}$$

Then the following theorem can be proved with the help of the Lie algorithm [3, 10, 12, 14].

Theorem 1. The system (13) is invariant with respect to the algebra with basis elements

$$P_1 = \partial_{x_1} \qquad P_2 = \partial_{x_2} \qquad P_3 = \partial_{v_2}$$

$$D = 2x_1\partial_{x_1} + x_2\partial_{x_2} + v^2\partial_{v^2} \tag{14}$$

$$Q = -x_1\partial_{x_1} + v^2\partial_{x_2} + v^1\partial_{v_1} \tag{15}$$

if $F = 1/\ln v^1$.

The operators P_1, P_2, P_3, and D correspond to generators of point transformations and Q corresponds to the operator

$$K = -t\partial_t + u_x\partial_x + \tfrac{1}{2}u_x^2\partial_u + u_t\partial_{u_t} \tag{16}$$

which is the generator of the contact transformations of the equation

$$u_t = \exp(1/u_{xx}). \tag{17}$$

The finite transformations corresponding to the infinitesimal operator (16) are given by the formulae

$$\tilde{t} = t\exp(-a) \qquad \tilde{x} = x + au_x \qquad \tilde{u} = u + \tfrac{1}{2}au_x^2 \qquad \tilde{u}_t = \exp(a)u_t. \tag{18}$$

Under these transformations u_{xx} is transformed as follows:

$$\tilde{u}_{xx} = \frac{u_{xx}}{1 + au_{xx}}. \tag{19}$$

Using (18) we can generate new solutions of equation (17) from known ones. Let $u = f(t, x)$ be a solution of equation (17). Thus, the solution of the first-order differential equation

$$u + \tfrac{1}{2}au_x^2 = f(\exp(-a)t, x + au_x) \tag{20}$$

which does not satisfy the equation

$$1 + au_{xx} = 0 \tag{21}$$

will be a solution of equation (17). Indeed, if the solution to be found satisfies equation (21) the change of variables (18) and (19) is singular. For example, we take the solution

$$u = \tfrac{1}{2}x^2 + et. \tag{22}$$

By applying the transformations (18) when $a = 2$ we obtain the equation

$$u = \tfrac{1}{2}x^2 + 2xu_x + u_x^2 + \mathrm{e}^{-1}t. \tag{23}$$

The general solution of equation (23) is given by formulae

$$u = -\tfrac{1}{4}x^2 + \mathrm{e}^{-1}t + c^2(t) \pm xc(t) \tag{24}$$

where c is arbitrary function of t, and

$$u = -\tfrac{1}{2}x^2 + \mathrm{e}^{-1}t. \tag{25}$$

The function u defined by (25) satisfies equation (17) and solution (24) satisfies equation (21), when $a = 2$ but does not satisfy equation (17). Thus, not every solution of equation (17) is transformed into another solution of this equation by finite transformations (18). This situation appears to be common in the case of contact transformations.

Nevertheless, we can use the operator Q to construct the ansatz reducing equation (17) to the system of two ordinary differential equations by applying the classic method of construction of invariant solutions [3, 10, 12, 14]. Thus, it is necessary to solve the invariant surface conditions

$$-x_1\frac{\partial v^1}{\partial x_1}+v^2\frac{\partial v^1}{\partial x_2}=v^1 \qquad -x_1\frac{\partial v^2}{\partial x_1}+v^2\frac{\partial v^2}{\partial x_2}=0. \tag{26}$$

By solving the corresponding characteristic equations

$$\frac{\mathrm{d}x_1}{-x_1}=\frac{\mathrm{d}x_2}{v^2}=\frac{\mathrm{d}v^1}{v^1}$$

we obtain three independent invariants

$$\omega_1=v^1x_1 \qquad \omega_2=\ln x_1+\frac{x_2}{v^2} \qquad \omega_3=v^2.$$

Then the solution of (26) is given implicitly by the invariant forms

$$v^1x_1=\varphi_1(v^2) \qquad \ln x_1+\frac{x_2}{v^2}=\varphi_2(v^2) \tag{27}$$

where φ_1 and φ_2 are arbitrary functions of v^2. From (27) we easily obtain the ansatz

$$v^1=\frac{\varphi_1(v^2)}{x_1} \qquad v^2=\frac{x_2}{(\varphi_2(v^2)-\ln x_1)} \tag{28}$$

corresponding to the operator Q.

Substituting (28) into (13) we obtain the system of ordinary differential equations

$$\ln\varphi_1-\varphi_2=v^2\frac{\mathrm{d}\varphi_2}{\mathrm{d}v^2} \qquad \frac{\mathrm{d}\varphi_1}{\mathrm{d}v^2}=v^2. \tag{29}$$

The general solution of system (29) has the form

$$\varphi_1=\tfrac{1}{2}((v^2)^2+C)$$

$$\varphi_2=\ln\frac{((v^2)^2+C)}{2e^2}+\frac{C_1}{v^2}+\frac{2C}{v^2}\int\frac{\mathrm{d}v^2}{(v^2)^2+C}.$$

Thus we obtain the system

$$u_t=\frac{(u_x)^2+C}{2t}$$

$$u_x=\frac{2x}{\ln[((u_x)^2+C)/2te^2]+(C_1/u_x)+(2\sqrt{C}/u_x)\arctan(u_x/\sqrt{C})} \tag{30}$$

when $C>0$ and e is the Euler number

$$u_t=\frac{(u_x)^2+C}{2t}$$

$$u_x=\frac{2x}{\ln[((u_x)^2+C)/2te^2]+(C_1/u_x)+(\sqrt{-C}/u_x)\ln[(u_x-\sqrt{-C})/(u_x+\sqrt{-C})]} \tag{31}$$

when $C<0$. To construct the solution of equation (17) it is necessary to integrate system (30) or (31). However, using the link between (3) and (4) it is easy to obtain the solution of equation (4) with $c(w)=-w^{-2}\exp(1/w)$ in the form

$$\exp\left(\frac{1}{w}\right)=\frac{(\theta)^2+C}{2t} \tag{32}$$

$$\theta=\frac{2x}{\ln[((\theta)^2+C)/2te^2]+(C_1/\theta)+(2\sqrt{C}/\theta)\arctan(\theta/\sqrt{C})} \tag{33}$$

when $C > 0$ and

$$\exp\left(\frac{1}{w}\right) = \frac{(\theta)^2 + C}{2t} \tag{34}$$

$$\theta = \frac{2x}{\ln[((\theta)^2 + C)/2te^2) + (C_1/\theta) + (\sqrt{-C}/\theta)\ln[(\theta - \sqrt{-C})/(\theta + \sqrt{-C})]} \tag{35}$$

where $C < 0$, respectively. Thus, (32) and (34) give two families of solutions for the nonlinear heat equation (4). In a similar way we can construct exact solutions of (4), where $c(w) = 1/r(w-1)^{(1/r)-1}/w^{(1/r)+1}$, $r \in \mathrm{R}$ and also $r \neq 0, \pm 1$.

It should be noted that using the ansätze of type (28) generated by some operators from (14), we obtain the solutions of equation (3) which are invariant with respect to the one-parameter group whose infinitesimal operator is $\alpha_1\partial_t + \alpha_2\partial_x + \alpha_3\partial_u + d(2t\partial_t + x\partial_x + 2u\partial_u) + \beta_1 x\partial_u$, where $\alpha_1, \alpha_2, \alpha_3, d, \beta_1$ are some constants. With the help of operator Q one can construct solutions which are not invariant with respect to any one-parameter subgroup of the point transformations group admitted by equation (3).

(2) Now we consider the nonlinear wave equation

$$u_{x_1x_2} = F(u, u_{x_1}, u_{x_2}). \tag{36}$$

The corresponding system is

$$v_2^1 + v_3^1 v^2 = F(x_3, v^1, v^2) \qquad v_1^2 + v_3^2 v^1 = F(x_3, v^1, v^2) \tag{37}$$

where $u_{x_1} \equiv v^1$ and $u_{x_2} \equiv v^2$.

Here we use the following definition of the conditional invariance [3, 15]. Let us consider the system of kth-order differential equations

$$L(x, u^i, \underset{1}{u^i}, \ldots, \underset{k}{u^i}) = 0 \tag{38}$$

where $x = (x_1, \ldots, x_n)$ denotes n independent variables, u^i denotes the dependent variables where $i = 1, 2, \ldots, m$, $\underset{k}{u^i}$ denotes the set of all kth-order partial derivatives of u^i with respect to x, and the infinitesimal operators

$$Q_A = \xi_A^l \partial_{x_l} + \eta_A^k \partial_{u^k} \qquad A = 1, 2, \ldots, N \tag{39}$$

forming a Lie algebra. (Summation under the repeated indices is understood.)

Definition 1. We shall say that system (38) is Q-conditionally invariant with respect to the operators Q_A from (39) if the following condition is fulfilled

$$\underset{k}{Q_A} L \Big|_{\substack{L=0 \\ Q_A u^i = 0}} \equiv 0 \tag{40}$$

where $Q_A u^i = 0$ is a set of equations

$$Q_A u^i = 0 \qquad DQ_A u^i = 0 \qquad \ldots \qquad D^p Q_A u^i = 0$$

$Q_A u^i = \xi_A^l \partial u^i/\partial_{x_l} - \eta_A^i$, D is the operator of total differentiation, and p is integer.

Defintition 2. System (38) is said to be conditionally invariant under the operators Q from (39), if there exist supplementary conditions on the solutions of (38) of the form

$$L_1(x, u^i, \underset{1}{u^i}, \ldots, \underset{k}{u^i}) = 0 \tag{41}$$

such that (38) together with (41) are Q-conditionally invariant under Q_A.

Then the following theorem can be proved.

Theorem 2. The system (37) is Q-conditionally invariant under the operators

$$Q_1 = \partial_{x_1}$$
$$Q_2 = \partial_{x_3} + \partial_{v^1} + F_1(v^1 - x_3)\partial_{v^2} \tag{42}$$

where F_1 is a smooth function, if $F = v^1 F_1(v^1 - x_3)$.

The correctness of theorem is easily verified with the help of the Q-conditional invariance criterion (40) [3, 15]. Note that Q_2 is not a prolongated Lie operator. The operators (42) lead to the ansatz

$$u_{x_1} = u + \varphi_1(x_2) \qquad u_{x_2} = F_1(\varphi_1(x_2))u + \varphi_2(x_2)$$

reducing (37) to the equation

$$\varphi_2(x_2) + \varphi_1'(x_2) = \varphi_1(x_2)F_1(\varphi_1(x_2)).$$

Having integrated the system

$$u_{x_1} = u + \varphi_1(x_2) \qquad u_{x_2} = F_1(\varphi_1(x_2))u + \varphi_1(x_2)F_1(\varphi_1(x_2)) - \varphi_1'(x_2)$$

one can obtain an exact solution of the wave equation

$$u_{x_1x_2} = u_{x_1}F_1(u_{x_1} - u)$$

in the form

$$u = -\varphi_1(x_2) + C\exp\left(x_1 + \int F_1(\varphi_1(x_2))\,\mathrm{d}x_2\right)$$

where $\varphi_1(x_2)$ is arbitrary function and C is a constant.

(3) Let us consider the equation

$$u_t - a(u)u_{xx} = u(1 - a(u)) \tag{43}$$

where $a(u)$ is a smooth function. In this case the corresponding system takes the form

$$v_x^1 + v_u^1 v^2 = v_u^2 v^1$$
$$v^1 - a(u)(v_x^2 + v_u^2 v^2) = u(1 - a(u)) \tag{44}$$

where $v^1 \equiv u_t$, $v^2 \equiv u_x$, v^1, v^2 are functions of indepedent variables x and u.

Theorem 3. The system (44) is conditionally invariant with respect to the operator

$$D = u\partial_u + v^1\partial_{v^1} + v^2\partial_{v^2} \tag{45}$$

under the side condition $v^1 = u$.

Theorem 3 is proved by means of a criterion of conditional invariance (see definition 2 [3, 15]). The operator (45) generates the ansatz

$$u_t = u\varphi_1(x) \qquad u_x = u\varphi_2(x)$$

which reduces system (44) to the system of ordinary differential equations

$$\varphi_1' = 0 \qquad \varphi_1 = 1 \qquad \varphi_2' + \varphi_2^2 = 1. \tag{46}$$

This sytem is compatible and its general solution has the form

$$\varphi_1 = 1 \qquad \varphi_2 = \frac{C\exp(2x) - 1}{C\exp(2x) + 1} \qquad C = \text{constant}.$$

Integrating the system

$$u_t = u \qquad u_x = u\frac{C\exp(2x)-1}{C\exp(2x)+1}$$

we obtain the solution of equation (43),

$$u = A\exp(x+t) + B\exp(t-x) \tag{47}$$

where A and B are arbitrary constants. The maximal invariance algebra of equation (43) is the two dimensional algebra with basic elements ∂_x and ∂_t. It is obvious that solution (47) does not correspond to the classical symmetry of equation (43).

(4) We apply this method to the equation

$$\frac{\partial^2 u}{\partial x_0^2} = F\left(\frac{\partial^2 u}{\partial x_0 \partial x_1}, \frac{\partial^2 u}{\partial x_1^2}\right) \tag{48}$$

where F is a smooth function.

One can obtain the Monge–Ampere equation setting $F = (\partial^2 u/\partial x_0\partial x_1)^2/(\partial^2 u/\partial x_1^2)$. We also recall that the nonlinear wave equation as well as the one-dimensional ideal fluid equation are mapped into equation (48) [21]. Using the invariance of equation (48) under the operators $\partial_u, x_0\partial_u, x_1\partial_u$, we write it in the form of the following system

$$\frac{\partial v^1}{\partial x_1} = \frac{\partial v^2}{\partial x_0} \qquad \frac{\partial v^3}{\partial x_0} = \frac{\partial v^2}{\partial x_1} \qquad v^1 = F(v^2, v^3) \tag{49}$$

where

$$\frac{\partial^2 u}{\partial x_0^2} \equiv v^1 \qquad \frac{\partial^2 u}{\partial x_0 \partial x_1} \equiv v^2 \qquad \frac{\partial^2 u}{\partial x_1^2} \equiv v^3.$$

Theorem 4. System (49) is invariant with respect to the continuous group of transformations with the infinitesimal operator

$$X = \xi^0(v^1, v^2, v^3)\partial_{x_0} + \xi^1(v^1, v^2, v^3)\partial_{x_1} \tag{50}$$

if ξ^0 and ξ^1 satisfy the system of linear equations

$$\xi_1^1 F_1 + \xi_2^1 - \xi_1^0 F_2 - \xi_3^0 = 0 \qquad \xi_2^0 F_2 = \xi_3^0 F_1 + \xi_1^1 F_2 + \xi_3^1 \tag{51}$$

where

$$\xi_a^k \equiv \frac{\partial \xi^k}{\partial v^a} \qquad F_a \equiv \frac{\partial F}{\partial v^{a+1}} \qquad k = 0, 1, a = 1, 2, 3.$$

Thus the invariance algebra of (49) is infinite-dimensional inspite of the fact that maximal invariance algebra of equation (48) is six-dimensional. The finite transformations

$$\tilde{x_0} = x_0 + a\xi^0 \qquad \tilde{x_1} = x_1 + a\xi^1 \tag{52}$$

correspond to the operator (50). Note that ξ^0 and ξ^1 depend on the second-order derivatives $\partial^2 u/\partial x_0^2$, $\partial^2 u/\partial x_0\partial x_1$, $\partial^2 u/\partial x_1^2$ in terms of the original variables. In the case of Lie–Bäcklund symmetry one can construct finite transformations in a closed form for point and contact symmetries only. We can apply the operator (50) to reduce equation (48) to the system of three ordinary differential equations for three unknown functions in the way used in the preceeding examples. With the help of transformations (52) new solutions of equation (48) can also be generated from a known solution. The existence of the infinite-dimensional invariance algebra of system (49) probably allows an exact linearization of this system by means of hodograph transformations

$$x_0 = x_0(v^2, v^3) \qquad x_1 = x_1(v^2, v^3). \tag{53}$$

Indeed, excluding v^1 from (49) we obtain the system

$$F_1 v_1^2 + F_2 v_1^3 = v_0^2 \qquad v_0^3 = v_1^2. \tag{54}$$

In terms of variables (53) this system is written in the form

$$F_1(v^2, v^3)\frac{\partial x_0}{\partial v^3} - F_2(v^2, v^3)\frac{\partial x_0}{\partial v^2} = -\frac{\partial x_1}{\partial v^3} \qquad \frac{\partial x_1}{\partial v^2} = \frac{\partial x_0}{\partial v^3}. \tag{55}$$

If we have a particular solution of the linear system (55) then it is easy to obtain an exact solution of equation (48). We can also formulate the nonlinear superposition principle for its solutions.

(5) Let us consider the system of two nonlinear equations

$$u_t = F(u_x, w_x) \qquad w_t = \Phi(u_x, w_x) \tag{56}$$

where F and Φ are arbitrary functions. The corresponding system is written in the form

$$v_2^1 = v_1^2 \qquad v_2^3 = v_1^4 \qquad v^1 = F(v^2, v^4) \qquad v^3 = \Phi(v^2, v^4) \tag{57}$$

where $u_t = v^1$, $u_x = v^2$, $w_t = v^3$, $w_x = v^4$, $t \equiv x_1$, $x \equiv x_2$. System (57) is invariant with respect to the continuous group of transformations with the infinitesimal operator

$$X = \xi^1(v^2, v^4)\partial_{x_1} + \xi^2(v^2, v^4)\partial_{x_2} \tag{58}$$

if ξ^1 and ξ^2 satisfy the system of linear equations

$$\xi_2^1 F_4 = \xi_4^1 F_2 + \xi_4^2 \qquad \xi_4^1 \Phi_2 = \xi_2^1 \Phi_4 - \xi_2^2 \tag{59}$$

where

$$\xi_a^k \equiv \frac{\partial \xi^k}{\partial v^a} \qquad F_a \equiv \frac{\partial F}{\partial v^a} \qquad \Phi_a \equiv \frac{\partial \Phi}{\partial v^a} \qquad k = 1, 2, a = 2, 4.$$

Inspite of the fact that ξ^1 and ξ^2 depend on v^2 and v^4, the operator (58) is not the operator of contact symmetry for the initial system (56).

System (57) can be linearized by means of hodograph transformations

$$x_1 = x_1(v^2, v^4) \qquad x_2 = x_2(v^2, v^4). \tag{60}$$

Indeed, excluding v^1 and v^3 from (57) we obtain the system

$$F_2 v_2^2 + F_4 v_2^4 = v_1^2 \qquad \Phi_2 v_2^2 + \Phi_4 v_2^4 = v_1^4.$$

In terms of variables (60) it is written in the form

$$\begin{aligned} &F_4(v^2, v^4)\frac{\partial x_1}{\partial v^2} - F_2(v^2, v^4)\frac{\partial x_1}{\partial v^4} = \frac{\partial x_2}{\partial v^4} \\ &\Phi_2(v^2, v^4)\frac{\partial x_1}{\partial v^4} - \Phi_4(v^2, v^4)\frac{\partial x_1}{\partial v^2} = \frac{\partial x_2}{\partial v^2}. \end{aligned} \tag{61}$$

(6) Now, we show that the conditional symmetry operator of the form

$$Q = \xi(t, x, u, v^1, v^2)\partial_u + \eta^1(t, x, u, v^1, v^2)\partial_{v^1} + \eta^2(t, x, u, v^1, v^2)\partial_{v^2} \quad (62)$$

which is admitted by the corresponding system can be used to construct the Bäcklund transformation for the original equation. In fact, if (62) is the operator of Q-conditional symmetry of the corresponding system, then we can construct the ansatz of type

$$u_t = F_1(t, x, u, \varphi_1, \varphi_2) \qquad u_x = F_2(t, x, u, \varphi_1, \varphi_2) \quad (63)$$

where φ_1 and φ_2 are unknown functions depending on two variables t and x, which reduces original equation to the system of two partial differential equations for φ_1 and φ_2. So, ansatz (63) maps a solution of the reduced system into a solution of the initial equation and can be used for constructing Bäcklund transformations. We present the example illustrating the above considerations.

Let us consider the equation

$$u_{xt} = [1 - k^2 u_x^2]^{1/2} \sin u. \quad (64)$$

The corresponding system is written in the form

$$\begin{aligned} &v_x^1 + v_u^1 v^2 = v_t^2 + v_u^2 v^1 \\ &v_t^2 + v_u^2 v^1 = \sqrt{1 - k^2 (v^2)^2} \sin u \end{aligned} \quad (65)$$

where $v^1 = u_t$, $v^2 = u_x$, v^1, v^2 are functions of three independent variables t, x, u. It can be proved that (65) is Q-conditionally invariant under the operator

$$Q = \partial_u + k \cos u\, \partial_{v^1} + k^{-1} \sqrt{1 - k^2 (v^2)^2}\, \partial_{v^2}. \quad (66)$$

Operator (66) generates the ansatz

$$u_x = k^{-1} \sin(u - \varphi_1) \qquad u_t = \varphi_2 + k \sin u \quad (67)$$

where φ_1 and φ_2 are unknown functions of two variables t and x (t and x are invariants of Q). Ansatz (67) reduces equation (64) to the system

$$\varphi_{2x} = \sin \varphi_1 \qquad \varphi_2 = \varphi_{1t}. \quad (68)$$

From (68) it follows that φ_1 satisfies the sine–Gordon equation

$$\varphi_{1tx} = \sin \varphi_1.$$

Using (68) one can rewrite (67) in the following form

$$u_x = k^{-1} \sin(u - w) \qquad u_t = w_t + k \sin u \quad (69)$$

where $\varphi_1 = w$. Thus, (69) gives the Bäcklund transformation which has been obtained in [22, 23] in another way. It maps the solution of the sine–Gordon equation $w_{xt} = \sin w$ into a solution of equation (64). Kruskall used this Bäcklund transformation in the discovery of an infinite sequence of the polynomial-conserved densities for the sine–Gordon equation.

Remark. In [18] Galaktionov proposed the method of 'nonlinear separation' in which the ansatz can be presented as a solution of linear ordinary differential equations [20]. In the general case the solution defined by the ansatz of type (2) does not satisfy a linear differential equation. Nevertheless, the Galaktionov ansatz can sometimes be obtained within the framework of this approach. Let us consider the ansatz [18]

$$u = w_1(t) + w_2(t) \cos x \quad (70)$$

which reduces the nonlinear heat equation

$$u_t = u_{xx} + u_x^2 + u^2 \tag{71}$$

to the system of two ordinary differential equations.

The system corresponding to (71) is written in the form

$$v_t^1 = v_u^2 v^2 + (v^2)^2 + u^2 \qquad v_u^1 = v_t^2 + v_u^2 v^1. \tag{72}$$

System (72) is conditionally invariant with respect to the operator

$$Q = v^2\partial_u + v^2(2v^1 - 2(v^2)^2 - 2u^2 + 2u - 1)\partial_{v^1} + (v^1 - (v^2)^2 - u^2)\partial_{v^2} \tag{73}$$

under the side condition

$$\tilde{v}_t^1 = 2(\tilde{v}^1)^2 + 2\tilde{v}^2$$

where $\tilde{v}^1$ and $\tilde{v}^2$ are the invariants of Q: $\tilde{v}^1 = v^1 + v^2 + 2uv^1 - u$, $\tilde{v}^2 = \sqrt{v^2 + 2uv^1 - u^2}$. The operator (73) generates the ansatz

$$u_t = \varphi_1 + \varphi_2(2u+1) - u \qquad u_x^2 = \varphi_1 + 2\varphi_2 u - u^2 \tag{74}$$

where $\varphi_1(t)$ and $\varphi_2(t)$ are unknown functions, which reduces equation (71) to the system of ordinary differential equations

$$\varphi_1' = 2\varphi_1\varphi_2 - 2\varphi_2^2 - 2\varphi_1 \qquad \varphi_2' = 2\varphi_2^2 + \varphi_1 + \varphi_2. \tag{75}$$

Having integrated the first equation of (74), one can obtain the ansatz

$$u(t,x) = \varphi(t)[\psi(t) + \theta(x)]$$

which is used in [18] for constructing explicit solutions of nonlinear evolution equations by the 'nonlinear separation' method.

It is easy to verify that (70) is the solution of the second equation from (74), provided that $w_2^2 - w_1^2 = \varphi_1$, $w_1 = \varphi_2$.

The fact that ansatz (70) reduces (71) to a system of ordinary differential equations is a straightforward consequence of the invariance of the two-dimensional functional subspace W_2 [19]. The reduction of (71) by the ansatz (74) is guaranteed by the existence of the symmetry operator Q of the corresponding system (72).

In addition we note that by using the Q-conditional symmetry operator of the corresponding system of type (5) and (6) we can also construct the group fibering of equation (3) also. Thus, we conclude that the suggested approach widens the applicability of the symmetry method to the construction of solutions of partial differential equations.

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